

Q Find the volume of the part of the ball $\rho \leq 2$ [$x^2 + y^2 + z^2 \leq 4$] that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

Soln The solid E can be expressed in spherical coordinates as :

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}\}$$

$$\begin{aligned} \text{Then, } \text{vol}(E) &= \iiint 1 \, dV = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho \\ &= [-\cos\phi]_{\pi/6}^{\pi/3} [\theta]_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^2 \\ &= \left[-\frac{1}{2} + \frac{\sqrt{3}}{2}\right] \cdot [2\pi] \cdot \left[\frac{8}{3}\right] \\ &= \frac{8\pi(\sqrt{3}-1)}{3} \end{aligned}$$